MATH 102:107, CLASS 33 (FRI NOV 24)

One end of a metal spring is attached to a wall, and the other to a mass which can move horizontally. Let $x(t)$ be the position of the mass at time t, so that

- $x < 0$ when the spring is compressed,
- $x > 0$ when the spring is stretched, and
- $x = 0$ when the spring is at its equilibrium length.

When the mass is moved from equilibrium and then let go, its position satisfies the differential equation

$$x'' = -kx$$

where k is a positive constant.

(1) Why is this a reasonable model? Does this make sense, based on your experience with springs?

Solution: The differential equation is saying that the spring exerts a force which opposes the displacement of the mass. I.e. if it is compressed, it will push, and if it is stretched, it will pull.

(2) Suppose that $k = 9$. Find a function of the form $x(t) = \sin(\omega t)$ or $x(t) = \cos(\omega t)$ which satisfies the differential equation, for some constant ω. (For a general value of k, what must ω be?)

Solution: Plug $\sin(\omega t)$ into the left side

$$(\sin(\omega t))'' = (\omega\cos(\omega t))' = -\omega^2\sin(\omega t)$$

Therefore, $x(t) = \sin(3t)$ is a solution. By similar reasoning, so is $x(t) = \cos(3t)$. In fact, any function of the form

$$x(t) = A\sin(3t) + B\cos(3t)$$

is a solution.

(3) Back to the case of $k = 9$. Suppose that the mass oscillates between $x = 2$ and $x = -2$, and $x(0) = 2$. Find $x(t)$.

Solution: The mass oscillates between -2 and 2, so it has **amplitude** 2. $x(0) = 2$ means that at $t = 0$, it is at its maximum value: therefore, it is a cosine curve. Since $k = 9$, it has **frequency** 3. So $x(t) = 2\cos(3t)$.

(4) What is $x(t)$ if the mass oscillates between $x = 2$ and $x = -2$, $x(0) = 1$, and $x'(0)$ is negative?

Solution: Again, we have amplitude 2 and frequency 3, but the fact that $x(0) = 1$ means that we have to shift our cosine curve by some **phase** P. That is,

$$x(t) = 2\cos(3(t + P))$$

for some P. Simply plugging in $t = 0$ and setting it equal to the initial condition allows us to find the phase.

$$1 = x(0) = 2\cos(3(0 + P)) \implies \cos(3P) = 1/2 \implies 3P = \pi/3 \implies P = \pi/9$$

Therefore, plugging in $P = \pi/9$, $x(t) = 2\cos(3(t+\pi/9))$, or $x(t) = 2\cos(3t+\pi/3)$. The fact that $x'(0)$ is negative allowed us to see that the graph of cosine should be shifted to the left, instead of to the right.

(5) (Hard) Suppose that the mass oscillates between $x = 2$ and $x = -2$, $x(0) = 1.2$, and $x'(0)$ is negative. Calculate $x\left(\frac{\pi}{6}\right)$.

Solution: We use the same approach

$$1.2 = x(0) = 2\cos(3P) \implies \cos(3P) = 0.6$$

$$\implies 3P = \arccos(0.6) \implies P = \frac{\arccos(0.6)}{3}$$

Thus, $x(t) = 2\cos\left(3(t + \frac{\arccos(0.6)}{3})\right) = 2\cos(3t + \arccos(0.6))$. Plugging in $t = \pi/6$,

$$x\left(\frac{\pi}{6}\right) = 2\cos\left(\frac{\pi}{2} + \arccos(0.6)\right)$$

For any θ, $\cos\left(\frac{\pi}{2} + \theta\right) = \sin(\theta)$, and so the above is equal to $2\sin(\arccos(0.6))$.

One can evaluate $\sin(\arccos(0.6))$ by looking at a right triangle with hypotenuse 1 and one sidelength of 0.6. $\arccos(0.6)$ is the angle adjacent to the 0.6 side, and therefore sine of it is the length of the other side. By the Pythagorean theorem, that's $\sqrt{1^2 - 0.6^2} = \sqrt{1 - 0.36} = \sqrt{0.64} = 0.8$. So $\sin(\arccos(0.6)) = 0.8$.

Therefore, $x\left(\frac{\pi}{6}\right) = 1.6$.